Stochastic Background of Gravitational Waves as a Benchmark for Extended Theories of Gravity

S. Capozziello,[1,*] M. De Laurentis,[2,†] and M. Francaviglia[3,‡]

[1] *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II" and INFN, Sez. di Napoli, Italy*
[2] *Politecnico di Torino and INFN Sez. di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy*
[3] *Dipartimento di Matematica, Università di Torino, and INFN Sez. di Torino, Italy*

The cosmological background of gravitational waves can be tuned by Extended Theories of Gravity. In particular, it can be shown that assuming a generic function $f(R)$ of the Ricci scalar R gives a parametric approach to control the evolution and the production mechanism of gravitational waves in the early Universe.

In the last thirty years several shortcomings came out in Einstein General Relativity (GR) and people began to investigate whether it is the only theory capable of explaining gravitational interactions. Such issues sprang up in Cosmology and Quantum Field Theory. In the first case, the Big Bang singularity, the flatness and horizon problems led to the result that the Standard Cosmological Model is inadequate to describe the Universe in extreme regimes. Besides, GR is a *classical* theory which does not work as a fundamental theory. Due to these facts and to the lack of a self-consistent Quantum Gravity theory, alternative theories have been pursued. A fruitful approach is that of Extended Theories of Gravity (ETGs) which have become a sort of paradigm based on corrections and enlargements of GR.

These theories have acquired interest in Cosmology owing to the fact that they "naturally" exhibit inflationary behaviors [1]. Recently, ETGs are playing an interesting role in describing the today observed Universe. In fact, the amount of good quality data of last decade has made it possible to shed new light on the effective picture of the Universe. In particular, the *Concordance Model* predicts that baryons contribute only for $\sim 4\%$ of the total matter - energy budget, while the exotic *cold dark matter* represents the bulk of the matter content ($\sim 25\%$) and the cosmological constant Λ plays the role of the so called "dark energy" ($\sim 70\%$). Although being the best fit to a wide range of data, the ΛCDM model is severely affected by strong theoretical shortcomings that have motivated the search for alternative models. Dark energy models mainly rely on the implicit assumption that GR is the correct theory of gravity indeed. Nevertheless, its validity on the larger astrophysical and cosmological scales has never been tested, and it is therefore conceivable that both cosmic speed up and dark matter represent signals of a breakdown in GR [2, 3, 4, 5, 6, 7, 8].

From an astrophysical viewpoint, ETGs do not require to find out candidates for dark energy and dark matter at fundamental level; the approach starts from taking into account only the "observed" ingredients (i.e. gravity, radiation and baryonic matter); this is in agreement with the early spirit of GR which could not act in the same way at all scales. In fact, GR has been definitively probed in the weak field limit and up to Solar System scales. However, a comprehensive effective theory of gravity, acting consistently at any scale, is far, up to now, to be found, and this demands an improvement of observational datasets and the search for experimentally testable theories. A pragmatic point of view could be to "reconstruct" the suitable theory of gravity starting from data. The main issues of this "inverse " approach is matching consistently observations at different scales and taking into account wide classes of gravitational theories where "ad hoc" hypotheses are avoided. In principle, the most popular dark energy models can be achieved by considering $f(R)$ theories of gravity and the same track can be followed to match galactic dynamics [9]. This philosophy can be taken into account also for the cosmological stochastic background of gravitational waves (GW) which, together with CMBR, would carry, if detected, a huge amount of information on the early stages of the Universe evolution.

[*] e-mail: capozziello@na.infn.it
[†] e-mail: mariafelicia.delaurentis@polito.it
[‡] e-mail: francaviglia@unito.it

In this paper, we face the problem to match a generic $f(R)$ theory with the cosmological background of GWs. GWs are perturbations $h_{\mu\nu}$ of the metric $g_{\mu\nu}$ which transform as 3-tensors. The GW-equations in the transverse-traceless gauge are

$$\Box h_i^j = 0\,. \tag{1}$$

Latin indexes run from 1 to 3. Our task is now to derive the analog of Eqs. (1) from a generic $f(R)$ given by the action

$$\mathcal{A} = \frac{1}{2k}\int d^4x\sqrt{-g}f(R)\,. \tag{2}$$

From conformal transformation, the extra degrees of freedom related to higher order gravity can be recast into a scalar field

$$\widetilde{g}_{\mu\nu} = e^{2\Phi}g_{\mu\nu} \qquad \text{with} \qquad e^{2\Phi} = f'(R)\,. \tag{3}$$

Prime indicates the derivative with respect to R. The conformally equivalent Hilbert-Einstein action is

$$\widetilde{\mathcal{A}} = \frac{1}{2k}\int \sqrt{-\widetilde{g}}d^4x\left[\widetilde{R} + \mathcal{L}\left(\Phi,\Phi_{;\mu}\right)\right] \tag{4}$$

where $\mathcal{L}\left(\Phi,\Phi_{;\mu}\right)$ is the scalar field Lagrangian derived from

$$\widetilde{R} = e^{-2\Phi}\left(R - 6\Box\Phi - 6\Phi_{;\delta}\Phi^{;\delta}\right)\,. \tag{5}$$

The GW-equation is now

$$\widetilde{\Box}\tilde{h}_i^j = 0 \tag{6}$$

where

$$\widetilde{\Box} = e^{-2\Phi}\left(\Box + 2\Phi^{;\lambda}\partial_{;\lambda}\right)\,. \tag{7}$$

Since no scalar perturbation couples to the tensor part of gravitational waves, we have

$$\widetilde{h}_i^j = \widetilde{g}^{lj}\delta\widetilde{g}_{il} = e^{-2\Phi}g^{lj}e^{2\Phi}\delta g_{il} = h_i^j \tag{8}$$

which means that h_i^j is a conformal invariant.

As a consequence, the plane-wave amplitudes $h_i^j(t) = h(t)e_i^j\exp(ik_mx^m)$, where e_i^j is the polarization tensor, are the same in both metrics. This fact will assume a key role in the following discussion.

In a FRW background, Eq.(6) becomes

$$\ddot{h} + \left(3H + 2\dot{\Phi}\right)\dot{h} + k^2a^{-2}h = 0 \tag{9}$$

being $a(t)$ the scale factor, k the wave number and h the GW amplitude. Solutions are combinations of Bessel's functions. Several mechanisms can be considered for the production of cosmological GWs. In principle, we could seek for contributions due to every high-energy process in the early phases of the Universe.

In the case of inflation, GW-stochastic background is strictly related to dynamics of cosmological model. This is the case we are considering here. In particular, one can assume that the main contribution to the stochastic background comes from the amplification of vacuum fluctuations at the transition between the inflationary phase and the radiation era. However, we can assume that the GWs generated as zero-point fluctuations during the inflation undergo adiabatically damped oscillations ($\sim 1/a$) until they reach the Hubble radius H^{-1}. This is the particle horizon for the growth of perturbations. Besides, any previous fluctuation is smoothed away by the inflationary expansion. The GWs freeze out for $a/k \gg H^{-1}$ and reenter

the H^{-1} radius after the reheating. The reenter in the Friedmann era depends on the scale of the GW. After the reenter, GWs can be detected by their Sachs-Wolfe effect on the temperature anisotropy $\triangle T/T$ at the decoupling. If Φ acts as the inflaton, we have $\dot{\Phi} \ll H$ during the inflation. Adopting the conformal time $d\eta = dt/a$, Eq. (9) reads

$$h'' + 2\frac{\chi'}{\chi}h' + k^2 h = 0 \tag{10}$$

where $\chi = ae^{\Phi}$. The derivation is now with respect to η. Inside the radius H^{-1}, we have $k\eta \gg 1$. Considering the absence of gravitons in the initial vacuum state, we have only negative-frequency modes and then the solution of (10) is

$$h = k^{1/2}\sqrt{2/\pi}\frac{1}{aH}C\exp(-ik\eta)\,. \tag{11}$$

C is the amplitude parameter. At the first horizon crossing ($aH = k$) the averaged amplitude $A_h = (k/2\pi)^{3/2}\,|h|$ of the perturbation is

$$A_h = \frac{1}{2\pi^2}C\,. \tag{12}$$

When the scale a/k becomes larger than the Hubble radius H^{-1}, the growing mode of evolution is constant, i.e. it is frozen. It can be shown that $\triangle T/T \lesssim A_h$, as an upper limit to A_h, since other effects can contribute to the background anisotropy. From this consideration, it is clear that the only relevant quantity is the initial amplitude C in Eq. (11), which is conserved until the reenter. Such an amplitude depends on the fundamental mechanism generating perturbations. Inflation gives rise to processes capable of producing perturbations as zero-point energy fluctuations. Such a mechanism depends on the gravitational interaction and then $(\triangle T/T)$ could constitute a further constraint to select a suitable theory of gravity. Considering a single graviton in the form of a monochromatic wave, its zero-point amplitude is derived through the commutation relations:

$$[h(t,x),\,\pi_h(t,y)] = i\delta^3(x-y) \tag{13}$$

calculated at a fixed time t, where the amplitude h is the field and π_h is the conjugate momentum operator. Writing the Lagrangian for h

$$\widetilde{\mathcal{L}} = \frac{1}{2}\sqrt{-\widetilde{g}}\widetilde{g}^{\mu\nu}h_{;\mu}h_{;\nu} \tag{14}$$

in the conformal FRW metric $\widetilde{g}_{\mu\nu}$, where the amplitude h is conformally invariant, we obtain

$$\pi_h = \frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{h}} = e^{2\Phi}a^3\dot{h} \tag{15}$$

Eq. (13) becomes

$$\Big[h(t,x),\,\dot{h}(y,y)\Big] = i\frac{\delta^3(x-y)}{a^3e^{2\Phi}} \tag{16}$$

and the fields h and $\dot{h}$ can be expanded in terms of creation and annihilation operators. The commutation relations in conformal time are

$$[hh'^{*} - h^{*}h'] = \frac{i(2\pi)^3}{a^3e^{2\Phi}}\,. \tag{17}$$

From (11) and (12), we obtain $C = \sqrt{2}\pi^2 He^{-\Phi}$, where H and Φ are calculated at the first horizon-crossing and, being $e^{2\Phi} = f'(R)$, the relation

$$A_h = \frac{H}{\sqrt{2f'(R)}}\,, \tag{18}$$

holds for a generic $f(R)$ theory. This is the central result of this paper and deserves some discussion. Clearly the amplitude of GWs produced during inflation depends on the theory of gravity which, if different from GR, gives extra degrees of freedom. On the other hand, the Sachs-Wolfe effect could constitute a test for gravity at early epochs. This probe could give further constraints on the GW-stochastic background, if ETGs are independently probed at other scales.

[1] A.A. Starobinsky, Phys. Lett. B, 91, 99 (1980).
[2] G. Magnano, M. Ferraris, M. Francaviglia, *Gen. Rel. Grav.* **19**, 465 (1987).
[3] S. Capozziello, *Int. Jou. Mod. Phys. D* **11**, 483 (2002).
[4] S. Nojiri and S. D. Odintsov, Phys. Lett. **B 576**, 5, (2003).
[5] G. Allemandi, A. Borowiec, and M. Francaviglia, *Phys. Rev. D* **70**, 043524 (2004).
[6] S. Nojiri and S. D. Odintsov, *Int. J. Geom. Meth. Mod. Phys.* **4**, 115 (2007).
[7] S. Capozziello and M. Francaviglia, *Gen. Rel. Grav.: Special Issue on Dark Energy* **40**, 357 (2008).
[8] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr - qc] (2008).
[9] S. Capozziello, V.F. Cardone and A. Troisi *Mon. Not. R. Astr.* **375**, 1423 (2007).